

Renormalization of Effective Hamiltonians

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Abstract

We introduce a way of implementing Wilson renormalization within the context of the theory of effective Hamiltonians. Our renormalization scheme involves manipulations at the level of the generalized G -matrix and is independent of any specific kinematics. We show how to calculate the beta function within this context and exhibit our method using simple scale-invariant quantum mechanical systems.

The theory of effective Hamiltonians and operators in many-body physics has a long and elaborate history [1]. This framework primarily involves manipulation of operators and is independent of the chosen kinematics. Our present effort is aimed at incorporating the concepts of renormalization within the context of the theory of effective Hamiltonians. Recently, other approaches for Hamiltonian renormalization have appeared both in field theory [2] and quantum mechanics [3,4].

Our general philosophy towards renormalization is inspired by the original work of Wilson [5]. Wilson's approach involved integrating out degrees of freedom above a certain momentum range to arrive at an effective action. This concept has primarily been implemented in the path integral formulation of a given problem. In recent years there has been a renewed effort to implement renormalization within a Hamiltonian formulation [2-4], as this may lead to a better understanding of key issues in strong interaction physics.

In this Letter we present a new non-perturbative scheme for renormalization which utilizes some of the more recent developments in the theory of effective Hamiltonians for many-body systems. Although our approach is quite general, we illustrate the methods with simple one-body quantum mechanical problems. In this way, we can exhibit the efficacy of our ideas without being overwhelmed by technical complications.

Consider the eigenvalue problem

$$H|\Psi_i\rangle = E_i|\Psi_i\rangle \quad i = 1, 2, \dots, n, \quad (1)$$

with n possibly infinite. There are very few Hamiltonians for which Eq. (1) is exactly solvable. In addition, the usual perturbation theory may turn out to be inadequate, and a large number of degrees of freedom associated with a many-body system may prevent a straightforward analysis of the problem. In realistic situations one would, in general, seek a subset of all the solutions of the above Hamiltonian. This naturally leads to the concept of an effective Hamiltonian.

We shall now briefly describe our construction of the effective Hamiltonian. We can arbitrarily split up any Hamiltonian H as

$$H = H_0 + V, \quad (2)$$

where H_0 is exactly solvable. One motivation for doing this step is to provide a convenient basis to work with, defined by the set of eigenvectors of H_0 :

$$H_0|\mu\rangle = E_\mu|\mu\rangle. \quad (3)$$

All matrices will be written with respect to this $|\mu\rangle$ basis, unless otherwise indicated.

We choose a model space \mathcal{M} which contains d basis vectors of H_0 . The operators P and Q which project into and out of our model space, respectively, are given by

$$\begin{aligned} P &= \sum_{\mu \in \mathcal{M}} |\mu\rangle\langle\mu|, \\ Q &= \sum_{\mu \notin \mathcal{M}} |\mu\rangle\langle\mu| = 1 - P. \end{aligned} \quad (4)$$

It is helpful to think of these projection operators as partitioned matrices :

$$\begin{aligned} P &= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right), \\ Q &= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array} \right). \end{aligned} \quad (5)$$

Consider, as in [6], a transformation of our Hamiltonian

$$\begin{aligned} \widetilde{H} &= e^{-S} H e^S, \\ |\widetilde{\Psi}_i\rangle &= e^{-S} |\Psi_i\rangle, \end{aligned} \quad (6)$$

where S is an operator to be determined shortly. It follows directly from this transformation that the eigenvalues of \widetilde{H} are the same as the eigenvalues of the original Hamiltonian, i.e.,

$$\widetilde{H}|\widetilde{\Psi}_i\rangle = E_i|\widetilde{\Psi}_i\rangle. \quad (7)$$

We will use the freedom in the choice of S to require that

$$\widetilde{H}(P|\widetilde{\Psi}_i\rangle) = E_i(P|\widetilde{\Psi}_i\rangle) \quad i = 1, 2, \dots, d. \quad (8)$$

As expressed in conventional applications to many-body problems, the goal of the effective Hamiltonian formalism is to construct an operator which acts only in a model space, yet gives us a subset of the exact eigenvalues of the full Hamiltonian. For our purposes here, we restate the goal as that of obtaining a subset of well-defined solutions of the eigenvalue problem for H .

For this purpose, we choose the effective Hamiltonian as

$$H_{\text{eff}} \equiv P\widetilde{H}P, \quad (9)$$

which clearly acts only on the states in the model space (which may, in itself, be infinite dimensional). From Eqs. (8) and (9) it also follows that

$$H_{\text{eff}}(P|\widetilde{\Psi}_i\rangle) = E_i(P|\widetilde{\Psi}_i\rangle). \quad (10)$$

The expression in Eq. (9) is therefore a consistent choice for H_{eff} .

The problem of finding H_{eff} now reduces to one of finding an appropriate S . Following [6,7], we choose to obtain S such that $S = QSP$. This implies that in our chosen basis S must have the form

$$S = \left(\begin{array}{c|c} 0 & 0 \\ \hline \hat{s} & 0 \end{array} \right) \quad (11)$$

where \hat{s} is a (presently) arbitrary $n - d$ by d matrix.

By choosing S in this particular way, S^n is zero for all $n > 1$. This leads to

$$e^S = 1 + S = \left(\begin{array}{c|c} 1 & 0 \\ \hline \hat{s} & 1 \end{array} \right) \quad (12)$$

with an analogous result for e^{-S} .

If we started out with an arbitrary hermitian Hamiltonian

$$H = \left(\begin{array}{c|c} a & b \\ \hline b^\dagger & f \end{array} \right). \quad (13)$$

with $a = a^\dagger$ and $f = f^\dagger$, then

$$\begin{aligned}\widetilde{H} &= e^{-S} H e^S \\ &= \left(\begin{array}{c|c} a + b\hat{s} & b \\ \hline -\hat{s}(a + b\hat{s}) + b^\dagger + f\hat{s} & f - \hat{s}b \end{array} \right)\end{aligned}\quad (14)$$

The effective Hamiltonian H_{eff} therefore takes the form

$$H_{\text{eff}} = P\widetilde{H}P = a + b\hat{s}. \quad (15)$$

We shall now exhibit an iterative method to obtain H_{eff} . Following [7] we define

$$Z \equiv H_{\text{eff}} - \omega = a + b\hat{s} - \omega, \quad (16)$$

which is equal to the effective Hamiltonian up to the arbitrary additive constant ω . Next we explicitly split H into $H_0 + V$ so that

$$\begin{aligned}H_0 &= \left(\begin{array}{c|c} \lambda_P & 0 \\ \hline 0 & \lambda_Q \end{array} \right), \\ V &= \left(\begin{array}{c|c} a - \lambda_P & b \\ \hline b^\dagger & f - \lambda_Q \end{array} \right),\end{aligned}\quad (17)$$

where λ_P And λ_Q are the (diagonal) matrices containing the eigenvalues of H_0 . Finally we introduce a generalized G -matrix defined as [6-7]

$$\begin{aligned}G(\omega) &\equiv PVP + PVQ \frac{1}{\omega - QHQ} QVP \\ &= PVP + PVQ \frac{1}{\omega - QH_0Q} QVP + \\ &\quad + PVQ \frac{1}{\omega - QH_0Q} QVQ \frac{1}{\omega - QH_0Q} QVP + \dots,\end{aligned}\quad (18)$$

which, for the above conventions, can be written as

$$G(\omega) = (a - \lambda_P) + b \frac{1}{\omega - f} b^\dagger. \quad (19)$$

Eqs. (16) and (18) can be solved iteratively to give Z and $G(\omega)$. From now on, we will denote $G(\omega)$ as G . One such iteration scheme is [6]

$$Z_1 = PH_0P + G - \omega P,$$

$$Z_n = \frac{1}{1 - G_1 - G_2 Z_{n-1} - G_3 Z_{n-2} Z_{n-1} - \cdots - G_{n-1} Z_2 Z_3 \cdots Z_{n-1}} Z_1, \quad (20)$$

where

$$G_k(\omega) = \frac{1}{k!} \frac{d^k}{d\omega^k} G(\omega). \quad (21)$$

H_{eff} can finally be constructed from the above solution for Z .

We note in passing that the generalized G -matrix may provide a leading approximation to H_{eff} . Within that approximation, our Eq. (18) bears resemblance to the effective Hamiltonian introduced in [3].

We shall now introduce the concept of renormalization within the above framework. We have seen above that the knowledge of the matrix G allows us to obtain Z , which is identical to H_{eff} up to an additive constant. In what follows, we shall therefore restrict our attention only to G . For the sake of convenience we choose to work in the momentum representation where the kinetic energy term in the Hamiltonian is diagonal. To introduce the concept of renormalization we shall focus our attention on the one-particle system. The formal generalization to a many-particle system would be straightforward. The matrix elements of G are here given by

$$G_{kk'} = \langle k | PVP | k' \rangle + \int dp dp' \langle k | PVQ | p \rangle \langle p | \frac{1}{\omega - QH_0Q} | p' \rangle \langle p' | QVP | k' \rangle + \cdots \quad (22)$$

Let us suppose that the potential V depends on a single coupling constant μ_0 , which we shall call the bare coupling constant. It is clear from Eq. (22) that the matrix element $G_{kk'}$ will be a function of μ_0 . The expression in Eq. (22) may, in general, require regularization due to the divergence arising from the integral. The regularization that we choose consists of introducing an ultraviolet cutoff Λ . The matrix element in Eq. (22) is now a function of the coupling constant μ_0 and the cutoff Λ . At the end of the calculation we must remove the cutoff, i.e. we must take Λ to ∞ , which, as discussed above, may in general lead to

divergence. One way to avoid the divergence is to replace the coupling constant μ_0 with a function of Λ , which we denote as $\mu(\Lambda)$, and then require that matrix element in Eq. (22) remain finite and independent of the cutoff as the cutoff is removed. In other words, we demand that

$$\lim_{\Lambda \rightarrow \infty} \frac{d}{d\Lambda} G_{kk'}(\Lambda, \mu(\Lambda)) = 0. \quad (23)$$

The function $\mu(\Lambda)$ thus plays the role of the renormalized coupling constant.

The dependence of the coupling constant on the cutoff is usually expressed in terms of the beta function, which is defined by

$$\beta(\mu) \equiv \Lambda \frac{d\mu}{d\Lambda}. \quad (24)$$

Within our formalism, Eqs. (23) and (24) can be used to calculate the beta function.

Note that once Eq. (23) is satisfied and $\mu(\Lambda)$ is determined, then H_{eff} (via Z), based on $G_{kk'}(\Lambda, \mu(\Lambda))$, should also be independent of Λ as $\Lambda \rightarrow \infty$. Thus, the complete problem of renormalization is solved.

We shall now illustrate the method prescribed above in two simple cases of a Dirac particle in 1 dimension and a Schrodinger particle in 2 dimensions [3,8]. In both these cases the interaction potential will be taken as a delta function in position space :

$$V(x) = -\mu_0 \delta^{(n)}(x), \quad (25)$$

where n is the dimension of configuration space. In the momentum space the interaction potential would simply be a constant, i.e.,

$$V(k) = -\mu_0. \quad (26)$$

We will choose H_0 to be the pure kinetic operator, and our model space to consist of all states with momenta less than λ . Thus Q projects onto the momentum range $[\lambda, \infty]$.

With the choice of the interaction potential described above, the series in Eq. (22) can be summed exactly and is given by

$$G_{kk'} = \frac{-\mu_0}{1 + \mu_0 I(\omega)} \delta(k - k'), \quad (27)$$

where $I(\omega)$ is given by

$$I(\omega) \equiv \int_{\lambda}^{\infty} d^n p \frac{1}{\omega - E_0(p)}. \quad (28)$$

Following the preceeding discussion we now introduce an ultraviolet cutoff Λ . Replacing μ_0 by the renormalized coupling constant μ and using Eqs. (23) and (24), we obtain the beta function as

$$\beta(\mu) = \mu^2 \Lambda \frac{\partial I}{\partial \Lambda}. \quad (29)$$

To obtain the explicit expression for the beta function we need to evaluate the integral appearing in Eq. (28). For the 1 dimensional Dirac particle we have $n = 1$, $E_0(p) = p + m$ and

$$I(\omega) \equiv \int_{\lambda}^{\Lambda} dp \frac{1}{\omega - (p + m)} = -\ln \left(\frac{\omega - (\Lambda + m)}{\omega - (\lambda + m)} \right). \quad (30)$$

The corresponding beta function is given by

$$\beta(\mu) = -\mu^2. \quad (31)$$

For the Schrodinger particle in 2 dimensions we have $n = 2$ and $E_0(p) = p^2/2$ (we set the mass of the particle as 1.) Proceeding exactly as before, we obtain

$$I(\omega) = -2\pi \ln \left(\frac{\omega - \Lambda^2}{\omega - \lambda^2} \right) \quad (32)$$

and

$$\beta = -4\pi\mu^2. \quad (33)$$

Note that the results in both examples above have the desirable property that the beta function is independent of the model space cutoff, λ . The beta functions calculated give rise to asymptotically free theories and generate the generally accepted pattern for the flow of the coupling constant for the two examples described above.

In conclusion, we have introduced a way of implementing a nonperturbative renormalization scheme within the context of many-body effective Hamiltonian theory. We have tested the method with applications to simple scale-invariant quantum mechanical systems. These examples exhibit the efficacy of our ideas and calculations for more realistic many-body systems and for quantum field theory are presently under investigation.

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